Mathematical Methods 3/4 Trial Exam 1 Solutions 2006
Free download and print from www.itute.com

1a. $f:(-1,\infty)\to R, f(x)=\dfrac{e^{x+1}}{2}$.

Equation of *f* is $y=\dfrac{e^{x+1}}{2}$. At $x=-1$, $y=\dfrac{e^0}{2}=\dfrac{1}{2}$; as $x\to\infty$, $y\to\infty$. $\therefore$range of *f* is $\left(\dfrac{1}{2},\infty\right)$.

Equation of f^{-1} is $x=\dfrac{e^{y+1}}{2}$, $2x=e^{y+1}$, $y=\log_e(2x)-1$,

$\therefore f^{-1}(x)=\log_e(2x)-1$.

1b. The domain of f^{-1} is the range of *f*, i.e. $\left(\dfrac{1}{2},\infty\right)$.

2a. Apply the product rule,

$$\frac{d}{dx}\left(2\sqrt{x}\cos(x)\right)=\left(\cos(x)\right)\left(\frac{1}{\sqrt{x}}\right)+\left(2\sqrt{x}\right)\left({}^{-}\sin(x)\right)$$

$$=\frac{\cos(x)-2x\sin(x)}{\sqrt{x}}.$$

2b. $f'(x)=\dfrac{1}{(2-x)^2}$,

$\therefore f(x)=\displaystyle\int\frac{1}{(2-x)^2}dx=\int(2-x)^{-2}dx=\frac{1}{2-x}+C$.

$f(3)=-1+C=3$, $\therefore C=4$, and $f(x)=\dfrac{1}{2-x}+4$.

3. To find the *x*-intercept, let $y=0$, $\therefore \sqrt{3}\sin(x)+\cos(x)=0$,

$\therefore \sqrt{3}\sin(x)=-\cos(x)$, $\dfrac{\sin(x)}{\cos(x)}=-\dfrac{1}{\sqrt{3}}$, $\tan(x)=-\dfrac{1}{\sqrt{3}}$,

$\therefore x=\dfrac{5\pi}{6}$ is the smallest positive value.

4a. $f(x)=2\sin(3x)$, where $x\in[0,2\pi]$,

$g(x)=\dfrac{1}{3}f\left(\dfrac{x}{2}\right)=\dfrac{1}{3}\left(2\sin 3\left(\dfrac{x}{2}\right)\right)=\dfrac{2}{3}\sin\left(\dfrac{3}{2}x\right)$. Amplitude $=\dfrac{2}{3}$ and period $=\dfrac{4\pi}{3}$.

4b. $h(x)=g(x)-\dfrac{2}{3}=\dfrac{2}{3}\sin\left(\dfrac{3}{2}x\right)-\dfrac{2}{3}$.

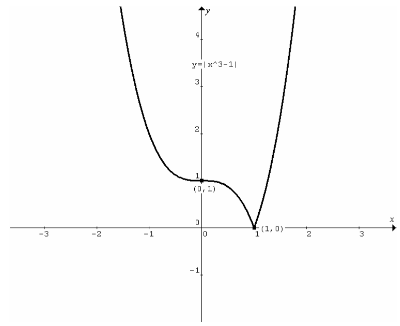
Domain: $[0,2\pi]$; range: $\left[-\dfrac{4}{3},0\right]$.

5. Solve simultaneously $y=ax-1$ and $y=x^2$.

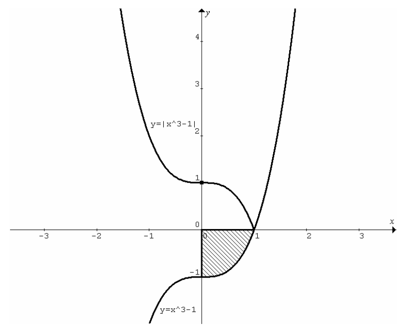
$x^2=ax-1$, $x^2-ax+1=0$. For $y=ax-1$ to be a tangent to $y=x^2$, $x^2-ax+1=0$ has only one solution,

i.e. $\Delta=0$, $(-a)^2-4(1)(1)=0$, $\therefore a=2$ given that $a>0$.

6a.



6b.



Area of the required region bounded by the two curves
= 2 × area of the shaded region (graph above)

$$=\left|2\int_0^1\left(x^3-1\right)dx\right|=\left|2\left[\frac{x^4}{4}-x\right]_0^1\right|=\left|-2\left(\frac{1}{4}-1\right)\right|=\frac{3}{2}.$$

6c. $y=\left|x^3-1\right|$ is differentiable for $x\neq 1$, $\therefore$maximal domain is $R\setminus\{1\}$.

7a. Given $u(x)=\log_e(x)$ and $f(x)=x^2+1$,

$\therefore f(u(x))=f(\log_e(x))=(\log_e(x))^2+1$.

7b. Apply the chain rule: $f'(u(x))=f'(u)\times u'(x)$

$=2u(x)\times\dfrac{1}{x}=\dfrac{2\log_e(x)}{x}$.

7c. Since $\dfrac{2\log_e(x)}{x}=f'(u(x))$, $\therefore \displaystyle\int\frac{2\log_e(x)}{x}dx=\int f'(u(x))dx$,

$\therefore \displaystyle\int\frac{\log_e(x)}{x}dx=\frac{1}{2}\int f'(u(x))dx=\frac{1}{2}f(u(x))+c=\frac{1}{2}(\log_e(x))^2+C$

An anti-derivative is $\dfrac{1}{2}(\log_e(x))^2$.

8. The water is pumped from the vessel at a constant rate of 2 m^3 per minute. $\therefore$ air is filling the vessel at a constant rate of 2 m^3 per minute.

Relationship between the height (h) of air above the water surface and the side-length of water surface (x):

$$\frac{x}{h}=\frac{3}{4},\ \therefore x=\frac{3h}{4}.$$

Volume of air above the water inside the vessel:

$$V=\frac{1}{3}x^2h=\frac{1}{3}\left(\frac{3h}{4}\right)^2h=\frac{3h^3}{16}.$$

Related rate: $\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}=\frac{9h^2}{16}\times\frac{dh}{dt}$.

When the depth of water is 1 m, the height of air above is 3 m,

$\therefore 2=\frac{9(3)^2}{16}\times\frac{dh}{dt}$, $\therefore \frac{dh}{dt}=\frac{32}{81}$, i.e. the height of air above is rising at $\frac{32}{81}$ m per minute. Hence the depth of water is falling at $\frac{32}{81}$ m per minute.

9a. Since $\mu=5$, $\therefore \Pr(X<5)=0.5$.

9b. Since $\mu=5$ and $\sigma=1.5$,

$\therefore \Pr(X\geq 8)=\Pr(X\geq\mu+2\sigma)=\frac{1}{2}(1-0.95)=0.025$.

10a. $f(x)=\begin{cases}0.3 & if\quad 0\leq x<2\\ p & if\quad 2\leq x\leq 6\\ 0 & if\ x<0 or x>6\end{cases}$

$0.3\times 2+p\times 4=1$, $4p=0.4$, $\therefore p=0.1$.

10b. $\Pr(-2<X\leq 3)$

$=\Pr(-2<X<0)+\Pr(0\leq X<2)+\Pr(2\leq X\leq 3)$

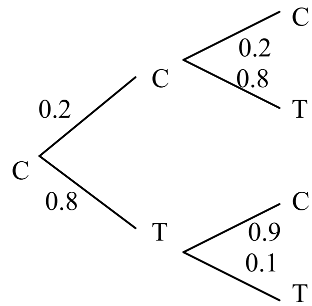
$=0+0.3\times 2+0.1\times 1=0.7$.

11. X has a binomial distribution with $n=3$, $p=\frac{1}{3}$ and

$\Pr(X=x)={}^3C_x\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{3-x}$.

x	0	1	2	3
$\Pr(X=x)$	$\frac{8}{27}$	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{1}{27}$

12.



$\Pr(3rdTEA)=\Pr(CCT)+\Pr(CTT)=0.2\times 0.8+0.8\times 0.1=0.24$.

Please inform mathline@itute.com re conceptual, mathematical and/or typing errors